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**BRIEF COMMUNICATIONS**

# Collisionless damping of short wavelength geodesic acoustic modes

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**Abstract**

Collisionless damping of geodesic acoustic mode (GAM) excited in the large safety factor ($q$) region of a tokamak plasma is investigated taking into account the effects of finite ion Larmor radius and guiding-center drift orbit width as well as parallel electric field contributions. A corresponding analytical expression for the damping rate including higher-order harmonics of ion transit resonances is systematically derived and agrees well with numerical results in its validity regime.

Geodesic acoustic mode (GAM) [1], predicted by Winsor *et al* in 1968 using the ideal magnetohydrodynamics (MHD) discription, is a toroidally symmetric and nearly poloidally symmetric mode peculiar to toroidal plasmas. Although GAM itself is, typically, linearly stable because of collisionless ion damping due to wave particle resonances in fusion plasmas [2, 3], GAM has recently attracted much attention, since it may be spontaneously excited by drift-wave turbulence via three-wave resonant parametric interactions [4–6] or external sources such as energetic particles [7, 8]. Furthermore, in the case of excitations by drift waves, due to its radially corrugated (zonal) structures, GAM in turn scatters drift wave from long-wavelength unstable domain to short-wavelength stable domain, and, thereby, provides self-regulation of drift-wave turbulence and, consequently, transport. As the thresholds for GAM excitations in terms of either the drift-wave intensity or the energetic particle energy density depend crucially on the GAM collisionless damping rate, it is desirable to have an analytical expression valid over a broad range of tokamak parameters, and this constitutes the motivation of this work.

Up to now, analytical theories for the GAM damping have assumed small magnetic drift orbits for the resonant ions, i.e. $k_r\rho_{\mathrm{d,res}} < 1$, such that only low-order harmonics of the transit resonances contribute. Here $k_r$ is the radial wave number, $\rho_{\mathrm{d,res}}$ is the resonant-ion magnetic drift orbit width, the subscript res denotes resonant particles, and $\rho_{\mathrm{d,res}} \approx q\rho_{\mathrm{i,res}}$ for an isotropic ion distribution assumed here, $\rho_{\mathrm{i,res}}$ is the resonant-ion Larmor radius. To be more specific, we note that, for GAM, the wave-particle resonance condition is given by

$$\omega = l\omega_{\mathrm{t,res}}$$

with $\omega_{\mathrm{t,res}} = v_{\parallel,\mathrm{res}}/qR_0$ being the transit frequency of the resonant circulating ions, $R_0$ being the tokamak major radius and $l$ = integers. For $k_r\rho_{\mathrm{d,res}} \approx k_r\rho_{\mathrm{i,res}}q < 1$ or, equivalently, $\omega_{\mathrm{d,res}} < \omega_{\mathrm{t,res}}$ with $\omega_{\mathrm{d,res}} \approx k_r v_{\mathrm{d,res}}$ being the resonant particle magnetic drift frequency, it can be readily demonstrated [2, 3] that the low-order harmonics (i.e. $l = \pm 1, \pm 2, \ldots$) transit resonances dominate. Taking $l = 1$ and noting $\omega \sim v_{\mathrm{it}}/R_0$ [1], we find

$$v_{\parallel,\mathrm{res}}/v_{\mathrm{it}} \sim q,$$

and $k_r\rho_{\mathrm{d,res}} \approx k_r\rho_{\mathrm{i,res}}q \sim k_r\rho_{\mathrm{it}}q^2$, $\rho_{\mathrm{it}}$ is the thermal ion Larmor radius. For $k_r\rho_{\mathrm{it}}q^2 \ll 1$, the corresponding collisionless GAM damping rate can then be straightforwardly derived and is given by [2, 9]

$$\gamma = -\frac{\sqrt{\pi}}{2}q\frac{v_{\mathrm{it}}}{R_0}\left[1+\frac{2(23/4+4\tau+\tau^2)}{q^2(7/2+2\tau)^2}\right]^{-1} \times \left\{\exp\{-\hat{\omega}_{\mathrm{G}}^2\}[\hat{\omega}_{\mathrm{G}}^4+(1+2\tau)\hat{\omega}_{\mathrm{G}}^2] + \frac{1}{4}(k_r\rho_{\mathrm{it}}q)^2\exp\{-\hat{\omega}_{\mathrm{G}}^2/4\} \times \left[\frac{\hat{\omega}_{\mathrm{G}}^6}{128}+\frac{1+\tau}{16}\hat{\omega}_{\mathrm{G}}^4+\left(\frac{3}{8}+\frac{7}{16}\tau+\frac{5}{32}\tau^2\right)\hat{\omega}_{\mathrm{G}}^2\right]\right\}, \quad (1)$$

in which

$$\hat{\omega}_{\mathrm{G}} = \frac{\sqrt{7+4\tau}}{2}q\left[1+\frac{2(23+16\tau+4\tau^2)}{q^2(7+4\tau)^2}\right]^{1/2} \quad (2)$$

is the real frequency of GAM normalized with $\omega_{\mathrm{tt}} = v_{\mathrm{it}}/(qR_0)$, transit frequency of thermal ions and $\tau = T_{\mathrm{e}}/T_{\mathrm{i}}$.

As $k_r\rho_{\mathrm{it}}q^2 \gtrsim 1$ or, equivalently, $\omega_{\mathrm{d,res}} \gtrsim \omega_{\mathrm{t,res}}$, higher-order ($|l| \gg 1$) harmonics of transit resonances can also contribute to significantly enhance the GAM collisionless damping rate [2, 3, 10], and no corresponding analytical expression exists due to the nontrivial task of summing up all the transit resonances. In this work, we adopt, instead, the $|\omega_{\mathrm{t,res}}/\omega_{\mathrm{d,res}}| \ll 1$ expansion in addition to the usual $1 \gg k_r\rho_{\mathrm{it}}$, $q \gg 1$ and $\tau \sim O(1)$ assumptions and derive a GAM dispersion relation which is valid for $k_r\rho_{\mathrm{it}}q^2 \gg 1$, i.e. $1 \gg k_r\rho_{\mathrm{it}}$, $k_r\rho_{\mathrm{it}}\tau^{1/2} \gg 1/q^2$. Note that the GAM dispersion relation in this parameter region is of practical importance, since equation (1) indicates that GAM typically exists in the $q \gg 1$ tokamak edge region in order to minimize the ion Landau damping [11]. Furthermore, nonlinear parametric excitation of GAM by drift waves such as the ion temperature gradient (ITG) instabilities also increases with $k_r\rho_{\mathrm{it}}$ [4]. We, therefore, have also kept $(k_r\rho_{\mathrm{it}})^2$ terms in the real frequency of GAM such that the derived dispersion relation is valid also for the short-wavelength kinetic GAM (KGAM) [4].

We consider a large aspect ratio axisymmetric tokamak with the equilibrium magnetic field given by $\boldsymbol{B}_0 = B_0(\boldsymbol{e}_\xi/(1+\epsilon\cos\theta) + (\epsilon/q)\boldsymbol{e}_\theta)$ , where $\xi$ and $\theta$ are, respectively, the toroidal and poloidal angles of the torus, $\epsilon = r/R_0 \ll 1$ is the inverse aspect ratio and $r$ is the minor radius. The perturbed particle distribution function $\delta f$ can be expressed as $\delta f = -eF_0\delta\phi/T + \exp[\mathrm{i}(m_{\mathrm{i}}c)/(eB^2)\boldsymbol{k}\times\boldsymbol{B}\cdot\boldsymbol{v}]\delta H_{\mathrm{g}}$, where the first term on the right-hand side is the adiabatic response to the scalar potential $\delta\phi$, and the second term, the non-adiabatic response, satisfies the linear gyrokinetic equation [12, 13]:

$$\left(\omega-\omega_{\mathrm{d}}+\mathrm{i}\omega_{\mathrm{t}}\frac{\partial}{\partial\theta}\right)\delta H_{\mathrm{g}} = \frac{e}{T}F_0J_0(k_\perp\rho_{\mathrm{L}})\omega\delta\phi, \quad (3)$$

where $\omega_t = v_\parallel/qR_0$ is the transit frequency, $\omega_d = \hat{\omega}_d \sin\theta = -k_r\rho_t v_t(v_\perp^2/2+v_\parallel^2)/(v_t^2 R_0)\sin\theta$ is the magnetic drift frequency associated with the geodesic curvature, $v_t$ and $\rho_t$ are thermal particle velocity and Larmor radius, $F_0$ is chosen to be Maxwellian at temperature $T$, $k_\perp$ is the perpendicular wavenumber and $k_\perp \approx k_r$ for GAM, $\rho_L = mcv_\perp/eB$ is the Larmor radius and $J_0(k_\perp\rho_L)$ is the Bessel function of zeroth index, describing finite Larmor radius effects (FLR). Subscripts to represent particle species are suppressed here for simplicity. Diamagnetic effects are negligible in our present ordering.

Ignoring finite electron Larmor radius and noting $\omega/\omega_{te} \sim \sqrt{m_e/m_i} \ll 1$, equation (3) can be readily solved for electrons, and we find $\delta f_e = e(\delta\phi - \overline{\delta\phi})F_{0e}/T_e$, where $\overline{\delta\phi}$ is the magnetic surface averaged $\delta\phi$. The quasi-neutrality condition thus becomes

$$\frac{e}{T_e}(\delta\phi - \overline{\delta\phi}) = -\frac{e}{T_i}\delta\phi + \langle J_0\delta H_{gi}/N_0\rangle, \tag{4}$$

where $N_0$ is the equilibrium density and $\langle\cdots\rangle$ indicates velocity space integration.

We now adopt the $1 \gg k_r\rho_{it} \gg 1/q^2$ and $\tau \sim O(1)$ orderings and consider the ions. In particular, we shall solve equation (3) separately for the non-resonant and resonant ions, i.e. $\delta H_g = \delta H_{g,nr} + \delta H_{g,res}$. Note that, since $\epsilon \ll 1$ and $|\omega|$ is much larger than the magnetic bounce frequency, we can ignore trapped-particle effects and assume constant $v_\parallel$. For the non-resonant ions, which involve the bulk thermal ions and determine the real frequency and the mode structures, we note that $|\omega_d/\omega| \sim |\omega_{dt}/\omega| \sim k_r\rho_{it} \ll 1$ and $|\omega_t/\omega| \sim |\omega_{tt}/\omega| \sim 1/q \ll 1$. Furthermore, note that the magnetic surface averaged $(n = 0, m = 0)$ potential $\overline{\delta\phi}$ of GAM is much larger than its poloidal variation, i.e. $|\overline{\delta\phi}| \gg |\widetilde{\delta\phi} \equiv \delta\phi - \overline{\delta\phi}|$. Taking $\delta H_{g,nr} = \delta H_{g,nr}^{(0)} + \delta H_{g,nr}^{(1)} + \delta H_{g,nr}^{(2)} + \cdots$ and $\delta\phi = \overline{\delta\phi} + \widetilde{\delta\phi}^{(1)} + \widetilde{\delta\phi}^{(2)} + \cdots$, the corresponding gyrokinetic equation for non-resonant ions can then be expanded in terms of the small parameters $k_r\rho_{it}$ and $1/q$ as

$$\delta H_{g,nr}^{(0)} = \frac{e}{T}F_0 J_0\overline{\delta\phi}, \tag{5}$$

$$\omega\delta H_{g,nr}^{(1)} - (\omega_d - i\omega_t\partial_\theta)\delta H_{g,nr}^{(0)} = \frac{e}{T}F_0 J_0\omega\widetilde{\delta\phi}^{(1)}, \tag{6}$$

$$\omega\delta H_{g,nr}^{(2)} - (\omega_d - i\omega_t\partial_\theta)\delta H_{g,nr}^{(1)} = \frac{e}{T}F_0 J_0\omega\widetilde{\delta\phi}^{(2)}, \tag{7}$$

and so on. Substituting $\delta H_{g,nr}^{(n)}$s into the quasi-neutrality condition, equation (4), we can readily derive

$$\widetilde{\delta\phi} = -\left[1 - b\left(\frac{3}{2}+\tau\right)\right]\tau\frac{\omega_{dt}}{\omega}\sin\theta\overline{\delta\phi} - \left[\frac{7}{4}+\tau - b\left(\frac{13}{4}+\frac{19}{4}\tau+2\tau^2\right)\right]\tau\frac{\omega_{dt}^2}{2\omega^2}\cos 2\theta\overline{\delta\phi}$$
$$- \left[\frac{9}{4}+\frac{7}{8}\tau\left(\frac{9}{4}+\frac{7}{4}\tau+\frac{1}{2}\tau^2\right)\cos 2\theta\right]\tau\frac{\omega_{dt}^3}{\omega^3}\sin\theta\overline{\delta\phi} - \left(\frac{\tau^2}{2}+\tau\right)\frac{\omega_{dt}\omega_{tt}^2}{\omega^3}\sin\theta\overline{\delta\phi}, \tag{8}$$

in which $b = k_r^2\rho_{it}^2/2$. Note again that $|\widetilde{\delta\phi}|/|\overline{\delta\phi}| \sim |\tau\omega_{dt}/\omega| \sim O(b^{1/2}\tau^{1/2}) \ll 1$ is assumed here. Meanwhile, solving for $\overline{\delta H_{g,nr}^{(4)}}$, we obtain the real part of GAM dispersion relation including FLR and finite guiding-center drift orbit width (FOW) [2] corrections,

$$D_r = b\left\{1 - \left(\frac{7}{4}+\tau\right)\frac{v_{it}^2}{\omega^2 R_0^2} + b\frac{v_{it}^2}{\omega^2 R_0^2}\left(\frac{31}{16}+\frac{9}{4}\tau+\tau^2\right) - \frac{v_{it}^4}{\omega^4 R_0^4 q^2}\left(\frac{23}{8}+2\tau+\frac{\tau^2}{2}\right)\right.$$
$$\left. - b\frac{v_{it}^4}{\omega^4 R_0^4}\left(\frac{747}{32}+\frac{481}{32}\tau+\frac{35}{8}\tau^2+\frac{1}{2}\tau^3\right)\right\}. \tag{9}$$

From equation (9), one finds the real frequency of GAM as

$$\omega_r = \left(\frac{7}{4}+\tau\right)^{1/2} \frac{v_{it}}{R_0} \left\{ 1 - \frac{b}{2}\left(\frac{31}{16}+\frac{9}{4}\tau+\tau^2\right)\left(\frac{7}{4}+\tau\right)^{-1} \right.$$
$$+\frac{b}{2}\left(\frac{747}{32}+\frac{481}{32}\tau+\frac{35}{8}\tau^2+\frac{\tau^3}{2}\right)\left(\frac{7}{4}+\tau\right)^{-2}$$
$$\left. +\frac{1}{2q^2}\left(\frac{23}{8}+2\tau+\frac{\tau^2}{2}\right)\left(\frac{7}{4}+\tau\right)^{-2} \right\}. \tag{10}$$

The $O(b)$ correction is important when one considers the mode conversion [14, 15] of GAM to short wavelength KGAM due to singular resonance associated with the GAM continuum spectrum [4]. Note that the expressions for $\widetilde{\delta\phi}$ and $\omega_r$ are valid as long as $1 \gg k_r\rho_{it}$, $1 \gg 1/q$ and $b\tau \ll 1$.

We now consider the resonant ions. Note that, for $k_r\rho_{it}q^2 \gg 1$, i.e. $k_r\rho_{d,res} \gg 1$, the relevant expansion parameter is then $|\omega_{t,res}/\omega_{d,res}| \sim |\omega_{t,res}/\omega| \ll 1$. Let

$$\delta H_{g,res} = \delta H^{(0)}_{g,res} + \delta H^{(1)}_{g,res} + \delta H^{(2)}_{g,res} + \cdots, \tag{11}$$

the linear gyrokinetic equation for resonant ions can then be expanded as

$$(\omega_d - \omega)\delta H^{(0)}_{g,res} = -\frac{e}{T_i} J_0 F_0 \omega \overline{\delta\phi}, \tag{12}$$

$$(\omega_d - \omega)\delta H^{(1)}_{g,res} = i\omega_t \frac{\partial}{\partial\theta}\delta H^{(0)}_{g,res} - \frac{e}{T_i} F_0 \omega \widetilde{\delta\phi}^{(1)}, \tag{13}$$

$$(\omega_d - \omega)\delta H^{(2)}_{g,res} = i\omega_t \frac{\partial}{\partial\theta}\delta H^{(1)}_{g,res} - \frac{e}{T_i} F_0 \omega \widetilde{\delta\phi}^{(2)}, \tag{14}$$

and so on. The imaginary part of the dispersion relation, meanwhile, is given by

$$D_i = \mathbb{I}\mathrm{m}\langle(J_0 \overline{\delta H_{g,res}})\rangle \Big/ \left(\frac{eN_0}{T_i}\overline{\delta\phi}\right). \tag{15}$$

The lowest order solution is obtained from equation (12) as

$$\delta H^{(0)}_{g,res} = -\frac{e}{T_i} J_0 F_0 \frac{\omega}{\omega_d - \omega} \overline{\delta\phi}. \tag{16}$$

Note that

$$\mathbb{I}\mathrm{m} \int \frac{d\theta}{2\pi} \frac{1}{\omega_d - \omega} = \int \frac{d\theta}{2\pi} \frac{\pi\delta(\theta-\theta_{res})}{\hat{\omega}_d \cos\theta}; \tag{17}$$

thus velocity space resonance would occur only if $|\hat{\omega}_d| \geqslant |\omega|$ or $(v_\perp^2/2 + v_\parallel^2)/v_{it}^2 \geqslant |\omega/\omega_{dt}|$, $\omega_{dt} = k_r\rho_{it}v_{it}/R_0$, $\theta_{res}$ is the point that satisfies the resonant condition $\omega = \hat{\omega}_d \sin\theta_{res}$ and the dominant contributions come around $\theta = -\sigma\pi/2$ with $\sigma \equiv \mathrm{sgn}(\omega/\omega_{dt})$. Expanding $(\omega - \hat{\omega}_d \sin\theta)^{-1}$ around $\theta = -\sigma\pi/2$, the straightforward integration gives

$$D_i^{(0)} = \omega \int d^3\vec{v} \frac{F_0 J_0^2/N_0}{|\hat{\omega}_d|^{1/2}[2(|\hat{\omega}_d| - |\omega|)]^{1/2}}. \tag{18}$$

The integration over the velocity space can be performed by dividing the velocity space into two parts, $|v_\parallel/v_{it}| \geqslant \Omega^{1/2}$ and $|v_\parallel/v_{it}| < \Omega^{1/2}$, in which $\Omega = \sigma\omega/\omega_{dt}$, and noting that wave-particle resonance occurs only when $(v_\perp^2/2+v_\parallel^2)/v_{it}^2 \geqslant \Omega$ and contributions are highly localized around $(v_\perp^2/2 + v_\parallel^2)/v_{\rm it}^2 = \Omega$, that is,

$$D_{\rm i}^{(0)} = 2\sqrt{\frac{2}{\pi}}\Omega\frac{\omega}{|\omega|}\left\{\int_0^{\Omega^{1/2}} {\rm d}y\int_{\sqrt{2(\Omega-y^2)}}^{\infty}{\rm d}x + \int_{\Omega^{1/2}}^{\infty}{\rm d}y\int_0^{\infty}{\rm d}x\right\}$$
$$\times\frac{(1-bx^2)\exp\{-x^2-y^2\}}{(x^2/2+y^2)^{1/2}(x^2/2+y^2-\Omega)^{1/2}}. \tag{19}$$

Noting that $|\Omega| \gg 1$ and including $O(b)$ terms, we have, after some tedious but straightforward algebra [4],

$$D_{\rm i}^{(0)} = \sqrt{2}\frac{\omega}{|\omega|}\left(1+\frac{\omega_{\rm dt}^2}{\omega^2}-2b\right)\exp\{-\sigma\omega/\omega_{\rm dt}\}. \tag{20}$$

Equation (20) corresponds to the $q \to \infty$ limit. We need to include higher-order corrections in order to obtain finite $q$ effects and connect with that given by equation (1). From equation (14), we get the first order correction to the perturbed distribution function:

$$\delta H_{\rm g,res}^{(1)} = \frac{\partial}{\partial\theta}\left(\frac{{\rm i}\omega_{\rm t}\omega S}{2(\omega_{\rm d}-\omega)^2}\right) - \frac{e}{T_{\rm i}}\frac{\omega F_0\widetilde{\delta\phi}^{(1)}}{\omega_{\rm d}-\omega}, \tag{21}$$

where $S = eF_0\overline{\delta\phi}/T_{\rm i}$. The first term on the r.h.s., which is the lowest order correction coming from finite $\omega_{\rm t}$, gives no contribution in the magnetic surface average. Meanwhile, the second term, the lowest order correction to $D_{\rm i}$ due to finite parallel electric field, can be again expanded around $\theta = -\sigma\pi/2$ to give

$$D_{\rm i}^{(1)} = \sqrt{2}\left(1-\sigma\frac{\omega_{\rm dt}}{2\omega}\right)\sigma\tau\frac{\omega_{\rm dt}}{|\omega|}\exp\{-\sigma\omega/\omega_{\rm dt}\}. \tag{22}$$

Note that $D_{\rm i}^{(1)}$ contains no $q$ dependence either. We therefore need to go to the next order. Substituting equation (21) into equation (14), the second order perturbed distribution function is

$$\delta H_{\rm g,res}^{(2)} = \frac{{\rm i}\omega_{\rm t}}{\omega_{\rm d}-\omega}\frac{\partial^2}{\partial\theta^2}\left(\frac{{\rm i}\omega_{\rm t}\omega S}{2(\omega_{\rm d}-\omega)^2}\right) - \frac{e}{T_{\rm i}}F_0\frac{\omega}{\omega_{\rm d}-\omega}\widetilde{\delta\phi}^{(2)}$$
$$+\frac{{\rm i}\omega_{\rm t}}{\omega_{\rm d}-\omega}\frac{\partial}{\partial\theta}\left(\frac{e}{T_{\rm i}}F_0\frac{\omega}{\omega_{\rm d}-\omega}\widetilde{\delta\phi}^{(1)}\right). \tag{23}$$

The first term is the second order correction from finite $\omega_{\rm t}$ and the second term is the second order correction from the finite parallel electric field. The last term, the lowest order finite $\omega_{\rm t}$ correction on the lowest order parallel electric field, can be written as a full derivative, and thus has no contribution. The second order correction to $D_{\rm i}$ is, thus, given by

$$D_{\rm i}^{(2)} = -\left\{\frac{1}{24\pi}\omega\frac{\partial^4}{\partial\omega^4}{\rm Im}\int{\rm d}^3\vec{v}\omega_{\rm t}^2\hat{\omega}_{\rm d}^2S\int_0^{2\pi}\frac{{\rm d}\theta}{2\pi}\frac{\cos^2\theta}{\omega_{\rm d}-\omega}\right.$$
$$\left.+\int{\rm d}^3v\int\frac{{\rm d}\theta}{2\pi}\frac{e}{T_{\rm i}}F_0\frac{\omega}{\omega_{\rm d}-\omega}\widetilde{\delta\phi}^{(2)}\right\}\Big/\left(\frac{eN_0}{T_{\rm i}}\overline{\delta\phi}\right). \tag{24}$$

Following the algebraic procedures in the lowest order calculations, $D_{\rm i}^{(2)}$ can be calculated and is given by

$$D_{\rm i}^{(2)} = \frac{\sqrt{2}}{24}\omega\omega_{\rm tt}^2\left(-\frac{4\sigma}{\omega_{\rm dt}^3}+\frac{\omega}{\omega_{\rm dt}^4}\right)\exp\left\{-\sigma\frac{\omega}{\omega_{\rm dt}}\right\}+\sqrt{2}\left(\frac{7}{4}+\tau\right)\frac{\omega_{\rm dt}^2}{\omega^2}\tau\exp\left\{-\sigma\frac{\omega}{\omega_{\rm dt}}\right\}. \tag{25}$$

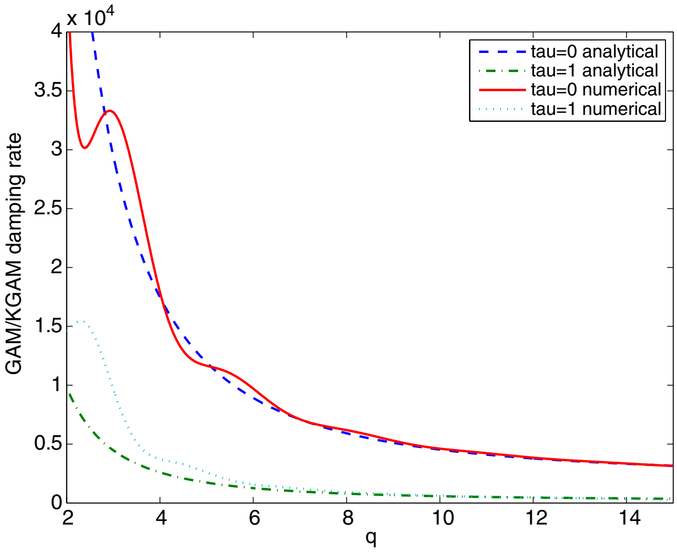


**Figure 1.** GAM/KGAM damping rate versus $q$. The four curves are, respectively, the numerical and analytical damping rates with $\tau = 0$ and 1.

(This figure is in colour only in the electronic version, see `www.iop.org`)

Note that $D_{\rm i}^{(2)}$ does contain the $1/q$ corrections via $\omega_{\rm tt}$ and thus we can stop the expansion here.

Finally, the collisionless damping rate of GAM with $O(b)$ and $O(1/q^2)$ corrections can be obtained from equations (20), (22), (25) and (9) and is given by

$$\frac{\gamma}{|\omega|} = -\frac{\sqrt{2}}{2b}\exp\left\{-\sigma\frac{\omega}{\omega_{\rm dt}}\right\}\left[1 + b\frac{v_{\rm it}^2}{\omega^2 R_0^2}\left(\frac{31}{16} + \frac{9}{4}\tau + \tau^2\right)\right.$$
$$\left. - 2b\frac{v_{\rm it}^4}{\omega^4 R_0^4}\left(\frac{747}{32} + \frac{481}{32}\tau + \frac{35}{8}\tau^2 + \frac{1}{2}\tau^3\right) - 2\frac{v_{\rm it}^4}{\omega^4 R_0^4 q^2}\left(\frac{23}{8} + 2\tau + \frac{\tau}{2}\right)\right]$$
$$\times\left\{1 + \frac{1}{24}\omega\omega_{\rm tt}^2\left(-\sigma\frac{4}{\omega_{\rm dt}^3} + \frac{\omega}{\omega_{\rm dt}^4}\right) + \sigma\frac{\omega_{\rm dt}}{\omega}\tau + \left(\tau^2 + \frac{5}{4}\tau + 1\right)\frac{\omega_{\rm dt}^2}{\omega^2} - 2b\right\}. \qquad (26)$$

We note, again, that this expression of the GAM damping rate is valid for $1 \gg k_r\rho_{\rm it} \gg 1/q^2$ and $b\tau \ll 1$.

We, now compare the analytical results derived here with the numerical results. The collisionless damping rate of GAM/KGAM versus $q$ is illustrated in figure 1 for the parameters $R_0 = 1.71$ m, $B_0 = 15$ T, $T_{\rm i} = 3$ KeV, $T_{\rm e} = \tau T_{\rm i}$, and $k_r\rho_{\rm it} = 0.1375$ with deuterium ions. These parameters are the same as those taken in the TEMPEST simulation [10]. The dashed and dotted–dashed curves are, respectively, our analytical results for $\tau = 0$ and $\tau = 1$; the solid and dotted curves are numerical results according to Gao and co-workers [3, 10] for $\tau = 0$ and $\tau = 1$, where a series of higher-order transit harmonics resonances are included to account for the FOW effect. In the $\tau = 0$ case, the numerical result is shown to agree excellently with the TEMPEST simulation [10]. From figure 1, we see that our analytical formula predicts damping rates in very good agreement with the numerical result in its validity regime; $q > (k_r\rho_{\rm it})^{-1/2} \sim 3$. Note that since the higher-order corrections in $\omega_r$ and $\gamma$ increase with $\tau = T_e/T_i$, the agreement between analytical and numerical results is, as expected, better for $\tau = 0$ than $\tau = 1$.

In conclusion, we have derived analytical expressions for the real frequency and damping rate of GAM, given, respectively, by equations (10) and (26). While the GAM real frequency expression, equation (10), is valid for 1, $\sqrt{T_i/T_e} \gg k_r\rho_{it}$ and $1 \gg 1/q$, the damping rate expression, equation (26), is valid for 1, $\sqrt{T_i/T_e} \gg k_r\rho_{it} \gg 1/q^2$ where higher-order harmonics of ion transit resonances contribute significantly to the GAM damping. Combining equation (26) with the previously derived damping rate expression, equation (1), valid for $1 \gg 1/q^2 \gg k_r\rho_{it}$ where only the low-order harmonics of ion transit resonances contribute, GAM dispersion relation can then be analytically determined over a broad range of 1, $\sqrt{T_i/T_e} \gg k_r\rho_{it}$ and $1 \gg 1/q$.

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