P249 W2005

Finch

(II) MHD Alfvén waves in Nonuniform Plasmas

Alfvén waves are fundamental e&m oscillations in magnetized plasmas; whether in laboratory or astrophysical/space envirnoments. Here, we would like to demonstrate that the MHD Alfvén waves take on peculiar properties due to nonuniformities, which, of course, are usually the rule rather than the exception.

(II.1) Theoretical Model with one-dimensional nonuniformities

We shall take the plasma to be cold and assume the dynamics satisfies the ideal MHD assumption. Thus, the eqns are

momentum conservation $\quad \rho_i \dfrac{d\underset{\sim}{u}}{dt} = \underset{\sim}{J} \times \underset{\sim}{B}/c$, $\quad \rho_i = m_i n_i$, Note $m_i \gg m_e$, $\frac{d}{dt} = \partial_t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}$

ideal MHD Ohm's Law $\quad \underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}/c = 0.$

Maxwell's eqn gives

Faraday's Law $\quad \underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\partial_t \underset{\sim}{B}$

Ampere's Law $\quad \underset{\sim}{\nabla} \times \underset{\sim}{B} = 4\pi \underset{\sim}{J}/c$.

For stationary equilibrium i.e, $\partial/\partial t = 0$, we have

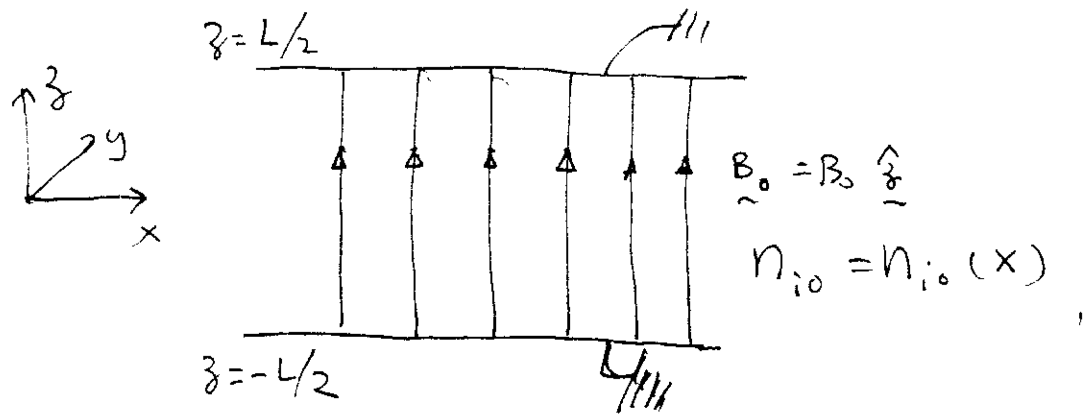
$$\underset{\sim}{U}_0 = 0, \quad \underset{\sim}{J}_0 = 0, \quad \underset{\sim}{B} = B_0 \ \hat{\underset{\sim}{z}}, \quad \underset{\sim}{E}_0 = 0.$$

ρ_{i0}, however, is arbitrary and, we shall take

$$\rho_{i0} = \rho_{i0}(x) \quad ;$$

that is, $n_{i0}(x)$, ion density, is inhomogeneous in x.

We are, thus, led to the following theoretical model



(II.2) <u>Linear Wave Equations</u>

We now consider infitesimally small linear perturbations of the above one-dimensional <u>slab</u> plasma. Thus, we have

$$\underset{\sim}{U} = \delta \underset{\sim}{U}, \quad \underset{\sim}{J} = \delta \underset{\sim}{J}, \quad \underset{\sim}{B} = \underset{\sim}{B}_0 + \delta \underset{\sim}{B}, \quad \underset{\sim}{E} = \delta \underset{\sim}{E},$$

$$\rho_i = \rho_{i0} + \delta\rho \quad .$$

Linearization corresponds to

$$|\delta\rho/\rho_{i0}|,\ |\delta \underset{\sim}{B}|/|\underset{\sim}{B}_0| \to 0^+ .$$

The corresponding ideal MHD equations are

$$\rho_{i0}(x)\,\partial_t \delta \underset{\sim}{u} = \delta \underset{\sim}{J} \times \underset{\sim}{B}_0/c$$

$$\delta \underset{\sim}{E} + \delta \underset{\sim}{u} \times \underset{\sim}{B}_0/c = 0 ,$$

$$-\frac{1}{c}\partial_t \delta \underset{\sim}{B} = \underset{\sim}{\nabla} \times \delta \underset{\sim}{E} ,$$

$$\frac{4\pi}{c}\delta \underset{\sim}{J} = \underset{\sim}{\nabla} \times \delta \underset{\sim}{B} .$$

It is more convenient (easier to picture) if we adopt the fluid displacement vector $\delta\underset{\sim}{\xi}$ such that

$$\delta \underset{\sim}{u} = \partial_t \delta \underset{\sim}{\xi} .$$

Combining ideal Ohm's law & Faraday's law, we find

$$-\frac{1}{c}\partial_t \delta \underset{\sim}{B} = \underset{\sim}{\nabla} \times (-\partial_t \delta \underset{\sim}{\xi} \times \underset{\sim}{B}_0/c)$$

$$\Rightarrow \quad \delta \underset{\sim}{B} = \underset{\sim}{\nabla} \times (\delta \underset{\sim}{\xi}_\perp \times \underset{\sim}{B}_0) ,$$

where we have denoted

$\delta \underset{\sim}{\xi} = \delta \underset{\sim}{\xi}_\perp + \delta\xi_\parallel \underset{\sim}{e}_\parallel$, where $\underset{\sim}{e}_\parallel = \hat{z}$ being the direction along $\underset{\sim}{B}_0$ and $\delta\underset{\sim}{\xi}_\perp$ being perpendicular to $\underset{\sim}{B}_0$.

Note that eqn of motion, we see $\delta\xi_\parallel = 0$.

Thus, letting

$$\delta \underset{\sim}{B} = \delta \underset{\sim}{B}_\perp + \delta B_\parallel \, \hat{\underset{\sim}{z}} \qquad , \text{ we find}$$

[Vector identity $\nabla \times (\delta \underset{\sim}{\xi}_\perp \times \underset{\sim}{B}_0) = \delta \underset{\sim}{\xi}_\perp (\underset{\sim}{\nabla} \cdot \underset{\sim}{B}_0) - \underset{\sim}{B}_0 (\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{\xi}_\perp)$
$+ (\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla}) \delta \underset{\sim}{\xi}_\perp - (\delta \underset{\sim}{\xi}_\perp \cdot \underset{\sim}{\nabla}) \underset{\sim}{B}_0$]

[and noting $\underset{\sim}{\nabla} \cdot \underset{\sim}{B}_0 = 0$, $\underset{\sim}{\nabla} \underset{\sim}{B}_0 = 0$, $\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{\xi}_\perp = \underset{\sim}{\nabla}_\perp \cdot \delta \underset{\sim}{\xi}_\perp$]

$$\delta \underset{\sim}{B}_\perp = (\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla}) \delta \underset{\sim}{\xi}_\perp$$

$$\delta B_\parallel = - B_0 (\underset{\sim}{\nabla}_\perp \cdot \delta \underset{\sim}{\xi}_\perp)$$

Eqn. of momentum conservation then reduces to

$$4\pi \rho_{i0}(x) \, \partial_t^2 \, \delta \underset{\sim}{u} = (\underset{\sim}{\nabla} \times \delta \underset{\sim}{B}) \times \underset{\sim}{B}_0$$

[Vector identity

$$\underset{\sim}{\nabla} (\delta \underset{\sim}{B} \cdot \underset{\sim}{B}_0) = \underset{\sim}{B}_0 \times (\underset{\sim}{\nabla} \times \delta \underset{\sim}{B}) + \delta \underset{\sim}{B} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}_0) + (\delta \underset{\sim}{B} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B}_0$$
$$+ (\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla}) \delta \underset{\sim}{B} \quad]$$

$$\Rightarrow \boxed{\begin{aligned} 4\pi \rho_{i0}(x) \, \partial_t^2 \, \delta \underset{\sim}{\xi}_\perp &= - \underset{\sim}{\nabla} (\delta \underset{\sim}{B} \cdot \underset{\sim}{B}_0) + (\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla}) \delta \underset{\sim}{B}_\perp + (\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla}) \delta B_\parallel \, \hat{\underset{\sim}{z}} \\ &= - \underset{\sim}{\nabla}_\perp (\delta B_\parallel B_0) + (\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla})^2 \delta \underset{\sim}{\xi}_\perp \end{aligned}}$$

Here, we have noted $\delta \underset{\sim}{B}_\perp$ in terms of $\delta \underset{\sim}{\xi}_\perp$

Defining $\delta b \equiv \delta B_{\parallel}/B_0$, $\nabla_{\parallel} = \left(\frac{\underset{\sim}{B_0}}{B_0}\cdot\underset{\sim}{\nabla}\right)$, we then have the desired set of eqns.

$$D_A(x)\,\delta\underset{\sim}{\xi}_{\perp} = \underset{\sim}{\nabla}_{\perp}\delta b \quad \text{and} \quad \delta b = -\underset{\sim}{\nabla}_{\perp}\cdot\delta\underset{\sim}{\xi}_{\perp} ;$$

where

$$D_A(x) \equiv \nabla_{\parallel}^2 - \frac{1}{V_A^2(x)}\partial_t^2 ,$$

and $V_A(x) = B_0/\sqrt{4\pi\rho_{i0}(x)}$ is the local Alfvén speed. $D_A(x)$ is, thus, the "local" shear Alfvén wave (SAW) operator.

Since the equilibrium plasma is uniform in y and z direction and time stationary, we can perform Fourier transform in y + z and Laplace transform in t. That is, we let

$$\delta\underset{\sim}{\xi}_{\perp}(\underset{\sim}{r},t) = \delta\hat{\underset{\sim}{\xi}}_{\perp}(x)\, e^{ik_y y + ik_{\parallel} z - i\omega t} + c.c. ,$$

we then have the following coupled set of o.d.e.'s,

$$\epsilon_A(x)\,\delta\hat{\underset{\sim}{\xi}}_{\perp}(x) = \underset{\sim}{\nabla}_{\perp}\delta\hat{b}(x) ,$$

$$\delta\hat{b}(x) = -\underset{\sim}{\nabla}_{\perp}\cdot\delta\hat{\underset{\sim}{\xi}}_{\perp}(x) ,$$

where $\quad \epsilon_A(x) = \omega^2/V_A^2(x) - k_\parallel^2$ is the "local" SAW dispersion relation, and

$$\underset{\sim}{\nabla}_\perp \equiv i k_y \hat{\underset{\sim}{y}} + \hat{\underset{\sim}{x}}\, d/dx \qquad d/dx \equiv d_x$$

Note we have three eqns for three independent variables, $\delta\hat{\underset{\sim}{\xi}}_\perp = \hat{\underset{\sim}{x}}\,\delta\hat{\xi}_x + \hat{\underset{\sim}{y}}\,\delta\hat{\xi}_y$ and $\delta\hat{b}$.

We now proceed to derive a single wave equation in terms of $\delta\hat{\xi}_x$. First, we note that

$$\epsilon_A\, \delta\hat{\xi}_y = i k_y\, \delta\hat{b} \quad ;$$

which, after being substituted into the $\delta\hat{b}$ eqn, leads to

$$\delta\hat{b} = -d_x \delta\hat{\xi}_x - i k_y\, \delta\hat{\xi}_y = -d_x\delta\hat{\xi}_x + \frac{k_y^2}{\epsilon_A}\,\delta\hat{b} \quad ,$$

or $\quad (k_y^2 - \epsilon_A)\,\delta\hat{b} = \epsilon_A\, d_x\, \delta\hat{\xi}_x \quad .$

Using $\delta\hat{b}$ expression in the $\hat{x}$-component of $\delta\hat{\underset{\sim}{\xi}}_\perp$, we then derive

$$\boxed{\frac{d}{dx}\, \frac{\epsilon_A}{k_y^2 - \epsilon_A}\, \frac{d}{dx}\,\delta\hat{\xi}_x(x) - \epsilon_A(x)\,\delta\hat{\xi}_x(x) = S(x,\omega) \quad ;}$$

where S denotes possible initial conditions via the Laplace transform. Meanwhile, the corresponding ~~wave~~ equation for the compressional Alfvén wave can be shown to be (Homework #1)

$$\left[\frac{d}{dx} \cdot \frac{1}{\epsilon_A} \frac{d}{dx} - k_y^2 \frac{1}{\epsilon_A} + 1 \right] \delta \hat{b}(x) = S_b(x, \omega).$$

(II.3) Singular solutions

Let us now examine the steady-state limit; i.e.

$$\omega = \omega_0 + i0^+ ,$$

and ignore the initial conditions (e.g. the perturbations are due to an external sources such as antennas). Here, 0^+ is needed to ensure causality; i.e. |perturbations| $\to 0$ as $t \to -\infty$.

The SAW wave eqn for $\delta\hat{\xi}_x$ then has a regular singular point at x_s where $\epsilon_{Ar}(x_s) = 0$; i.e

$$\omega_0^2 = k_\parallel^2 V_A^2(x_s).$$

To examine the singular solutions at $x = x_s$, we shall perform the following Frobenius expansion; that is

$$\delta\hat{\xi}_x = \delta\hat{\xi}_x^0 + \delta\hat{\xi}_x^1 + \cdots, \quad \text{where}$$

adopting $t = x - x_s$, we have for $n \geq 1$

$$\lim_{|t|\to 0} |\delta\hat{\xi}_x^{(n)}| / |\delta\hat{\xi}_x^{(0)}| \Rightarrow |t|^n \Rightarrow 0.$$

The wave eqn then becomes

$$\frac{d}{dt}\, t\, \frac{d}{dt}\delta\hat{\xi}_x - t\, k_y^2\, \delta\hat{\xi}_x = 0.$$

we are only interested in the singular solution; thus,

$$O(1) \qquad \frac{d}{dt}\, t\, \frac{d}{dt}\delta\hat{\xi}_x^0 = 0 \quad \Rightarrow \quad \delta\hat{\xi}_x^0 = C_0 \ln t$$

$$O(t) \qquad \frac{d}{dt}\, t\, \frac{d}{dt}\delta\hat{\xi}_x^1 = t\, k_y^2\, \delta\hat{\xi}_x^0 = C_0 k_y^2\, t \ln t$$

$$\Rightarrow \quad \delta\hat{\xi}_x^1 = C_0 k_y^2 \frac{t^2}{4} \ln t + (\text{h.o.t.})$$

higher-order-terms

we can keep on going on. It's enough for now, so

$$\delta\hat{\xi}_{xs} = C_0 \ln t \left[1 + (k_y^2/4)\, t^2 + (\text{h.o.t.})\right] ..$$

Subscript s denotes the singular branch. $\delta\hat{\xi}_x$ thus has a logarithmic singularity at $x = x_s$. With $\delta\hat{\xi}_{xs}$ obtained, we can derive the correspondi δb from

$$\frac{d\delta\hat{b}}{dx} = \epsilon_A(x)\,\delta\hat{\xi}_x \simeq \epsilon_A' t\,\delta\hat{\xi}_{xs} ,$$

We then find, by straightforward integration by parts,

$$\delta\hat{b}(x) = b_0 + c_0\,\epsilon_A' \frac{t^2}{2}\ln t + (h.o.t.)$$

$\delta\hat{b}(x)$ thus has much weaker singular behaviours. Finally, let us find $\delta\hat{\xi}_y$; which is given by,

$$\epsilon_A\,\delta\hat{\xi}_{ys} \simeq \epsilon_A' t\,\delta\hat{\xi}_{ys} = i k_y\,\delta\hat{b} \simeq i k_y b_0 \; ; \text{ i.e.}$$

$$\delta\hat{\xi}_{ys} \simeq i k_y b_0 / \epsilon_A' t .$$

The two arbitrary constants, c_0 and b_0, meanwhile are constrained by the relationship

$$\delta\hat{b} = -\underline{\nabla}_\perp \cdot \delta\hat{\underline{\xi}}_\perp \simeq \left(k_y^2 b_0/\epsilon_A' t\right) - (c_0/t) + (h.o.t.)$$

Equating the $(1/t)$ term then leads to

$$c_0 = b_0 k_y^2 / \epsilon_A'$$

We thus have demonstrated that the singular solutions, $\delta\hat{\xi}_{xs}, \delta\hat{\xi}_{ys}$ are coupled to the CAW, $\delta\hat{b} \simeq b_0$. In otherwords, as a compressional Alfvén wave with $\omega = \omega_0$ propagates across a nonuniform plasma, it excited large-amplitude (singular) SAW with $\underset{\sim}{\nabla}_\perp \cdot \delta\hat{\underset{\sim}{\xi}}_{\perp s} \simeq 0$ at $x = x_s$ where ω_0 matches the local SAW frequency; i.e., $\omega_0 = k_\parallel V_A(x_s) = \omega_A(x_s)$. Note also that $|\delta\hat{b}_{ys}| \propto |\delta\hat{\xi}_{ys}| \sim |1/t| \gg |\delta\hat{b}_{xs}| \propto |\delta\hat{\xi}_{xs}| \sim 1/|\ln t|$. Finally, we remark that $\delta\hat{b}(x)$ solution can also be directly derived by solving the CAW wave equation.

(II.4) Collisionless resonant wave-energy absorption

In this subsection, we want to demonstrate that the singular behaviours of SAW at $x = x_s$ can lead to wave-energy absortion even in the "collisionless" limit; this is very much similar to the wave energy absorption of Landau damping. The major difference is Landau damping occurs in the velocity space; while SAW resonant absorption occurs in the configuration space.

To calculate the resonant absorption rate, we note that, in the steady state, the rate of the total particle kinetic energy change is given by

$$\frac{d}{dt}W = \frac{1}{2}\,\mathrm{Real}\int d^3\underset{\sim}{x}\;(\delta\hat{\underset{\sim}{J}}^{*}\cdot\delta\hat{\underset{\sim}{E}}).$$

Noting that $\underset{\sim}{\nabla}\times\delta\hat{\underset{\sim}{B}} = \frac{4\pi}{c}\,\delta\hat{\underset{\sim}{J}}$, we have, with

$\partial_t|\delta\hat{\underset{\sim}{B}}|^2 = 0$,

$$\frac{dW}{dt} = -\frac{1}{2}\int d^3\underset{\sim}{x}\;\mathrm{Real}\;\underset{\sim}{\nabla}\cdot\underset{\sim}{P}_{EB}\,,\quad\text{where}$$

$\underset{\sim}{P}_{EB} = \frac{c}{4\pi}\,\delta\underset{\sim}{E}\times\delta\underset{\sim}{B}^{*}$ is the Poynting vector. Since system is uniform in y and z, only $\frac{dP_{EBx}}{dx}$ contributes; i.e.,

$$\frac{dW}{dt} = -\frac{1}{2}\,L_y L_z\int_{x_1}^{x_2} dx\,\frac{d}{dx}\,\frac{c}{4\pi}\,\mathrm{Real}\,(\delta\hat{E}_y\,\delta\hat{B}_z^{*}).\qquad x_1\le x\le x_2$$

Now $\delta\hat{E}_y = -i\omega_0\frac{B_0}{c}\,\delta\hat{\xi}_x$, thus, we have

$$\frac{dW}{dt} = \frac{1}{8\pi}\,\omega_0 B_0 L_y L_z\int_{x_1}^{x_2} dx\,(-1)\,\mathrm{Im}\,\frac{d}{dx}(\delta\hat{\xi}_x\,\delta\hat{B}_{\parallel}^{*})$$

$$\simeq -\frac{1}{8\pi}\,\omega_0 B_0 L_y L_z\int_{x_1}^{x_2} dx\,\mathrm{Im}\left(\delta\hat{B}_{\parallel}^{*}\,\frac{d\delta\hat{\xi}_x}{dx}\right).$$

Note $\partial_x\delta\hat{\xi}_{xs} \cong -ik_y\delta\hat{\xi}_{ys} \cong b_0 k_y^2/\epsilon_A$, $\delta\hat{B}_{\parallel}^{*} \approx b_0^{*}$

we obtain

$$\frac{dW}{dt} = -\frac{1}{8\pi}\omega_0 B_0 L_y L_z k_y^2 \int_{x_1}^{x_2} dx \, |b_0|^2 \, \mathrm{Im}\left(\frac{1}{\epsilon_A}\right).$$

Note $\epsilon_A = \frac{\omega^2}{V_A^2} - k_\parallel^2 \cong \frac{\omega_0^2 + 2i\omega_0 0^+}{V_A^2} - k_\parallel^2$, thus,

$$\mathrm{Im}\,\frac{1}{\epsilon_A} = \begin{cases} 0 & \text{for } \omega_0^2 \neq k_\parallel^2 V_A^2 \\ -\infty & \text{for } \omega_0^2 = k_\parallel^2 V_A^2 \end{cases}.$$

In fact, we can neatly express, for $x = x_{s_j}$ where $\epsilon_A(x_{s_j}) = 0$, $j = 1, 2, \ldots$

$$\mathrm{Im}\,\frac{1}{\epsilon_A} = -\pi\,\delta(\epsilon_A) = -\pi \sum_j \frac{1}{|\epsilon_A'|_{x_{s_j}}}\,\delta(x - x_{s_j})$$

Hence, finally, we arrive at

$$\frac{dW}{dt} = \frac{1}{8}\omega_0 B_0 L_y L_z k_y^2 \sum_{j=1}^{N} |b_0|^2_{x=x_{s_j}} \frac{1}{|\epsilon_A'|_{x_{s_j}}}.$$

(II.5) Shear Alfvén Continuum and phase Mixing

In the preceding lecture, we discussed the singular solution and the associated "collisionless" wave-energy absorption when the frequency of a steady-state compressional Alfvén wave drive matches the local SAW frequency.

We now take the alternative perspective; i.e., that of an initial-value approach. Specifically, we would like to find the temporal ~~respo~~ dynamics subject to an impulsive excitation at $t=0$. Specifically, we are interested $\delta\hat{\underset{\sim}{\xi}}_\perp(x,t)$ for $t>0$ subject to

$$\delta\hat{b}(x,t) = b(x)\,\delta(t),$$

and the initial conditions $\delta\hat{\underset{\sim}{\xi}}_\perp(x,t)=0$ for $t\le 0$.

Again, we have

$$D_A(x,t)\,\delta\hat{\underset{\sim}{\xi}}_\perp(x,t) = \underset{\sim}{\nabla}_\perp\delta\hat{b},$$

with D_A, however, takes the following form

$$D_A = -\left[k_\parallel^2 + \frac{1}{V_A^2(x)}\frac{\partial^2}{\partial t^2}\right].$$

Now, from preceding lectures, $\delta\hat{b} \cong b_0\delta(t)$, i.e., $\delta\hat{b}(x,t)$ varies much slowlier in x than $\delta\hat{\xi}_\perp$ at least time asymptotically. We, thus, have, as $t\to\infty$,

$$D_A(x,t)\,\delta\hat{\xi}_y(x,t) = i\,k_y\,b_0^{(x)}\delta(t)$$

for $t>0$, we have

$$\delta\hat{\xi}_y(x,t) = A_1(x)\,e^{i\omega_A(x)t} + A_2(x)\,e^{-i\omega_A(x)t}.$$

That $\delta\hat{\xi}_y = 0$ at $t=0 \Rightarrow A_1 = -A_2$. or

$$\boxed{\delta\hat{\xi}_y(x,t) = i\,A(x)\sin[\omega_A(x)t]}.$$

Note $\omega_A(x) = k_\parallel V_A(x) > 0$ w/o loss of generality. To determine $A(x)$, we perform $\int_{0_-}^{0_+} dt$ and find

$$\frac{1}{V_A^2(x)}\frac{\partial}{\partial t}\delta\hat{\xi}_y\Big|_{0_-}^{0_+} = i\,k_y\,b_0(x)$$

Note $\delta\hat{\xi}_y = 0$ for $t\le 0^-$, we have

$$\frac{i}{V_A^2}A(x)\,\omega_A(x)\cos[\omega_A(x)t]\Big|_{0_+} = i\,k_y\,b_0(x)\;;$$

or

$$A(x) = k_y b_0(x)\,V_A^2(x)/\omega_A(x).$$

All together, we have

$$\delta \hat{\xi}_y(x,t) = i \frac{k_y b_0(x) V_A^2(x)}{\omega_A(x)} \sin[\omega_A(x) t].$$

Note that the oscillation frequency is the local SAW frequency, $\omega_A(x)$. Since $\omega_A(x)$ varies smoothly in x, the frequency spectrum is thus continuously varying in x ~~; i.e.~~ and, hence, the statement that in nonuniform plasmas, SAW has a continuous spectrum or continuum.

Now as to $\delta \hat{\xi}_x(x,t)$, since in the steady-state limit, we have noted that $\underset{\sim}{\nabla}_\perp \cdot \delta \underset{\sim}{\xi}_\perp \simeq 0$, we, therefore, expect that, as $t \to \infty$, we should have

$$\frac{d}{dx} \delta \hat{\xi}_x \simeq -i k_y \delta \hat{\xi}_y = b_0 \frac{k_y^2 V_A^2(x)}{\omega_A(x)} \sin[\omega_A(x) t].$$

Since $d\omega_A/dx \neq 0$, the dominant x variation, as $t \to \infty$, enters via the $\omega_A(x) t$ term, it is then easy to show that, as $t \to \infty$,

$$\delta \hat{\xi}_x(x,t) \Rightarrow b_0 \frac{k_y^2 V_A^2(x)}{\omega_A(x)} \cdot \frac{(-1)}{\omega_A' t} \cos[\omega_A(x) t].$$

Note that the $(1/t)$ dependence comes from the observation that each location has different oscillation frequency $\omega_A(x)$ and $d\omega_A(x)/dx \neq 0$; and, hence, $\delta\hat{\xi}_x$, the displacement in the x or non-uniform direction, will ~~not~~ involve coupling between neighboring SAW oscillations and, thus, temporally decay as $|1/t|$ due to the phase mixing. Meanwhile, as $t \to \infty$,

$|\delta\hat{b}_x|/|\delta\hat{b}_y| \sim |\delta\hat{\xi}_x|/|\delta\hat{\xi}_y| \sim |1/t| \ll 1$.

Attached here are magnetic fluctuations measured by the AMPTE/CCE satellite in the Earth's magnetosphere (dayside, outbound-prenoon and inbound-late afternoon) δB_R corresponds to $\delta\hat{b}_x$ (the nonuniform radial direction) and δB_E ~~the~~ to $\delta\hat{b}_y$ (the East-West azimuthal symmetry direction). Shown here clearly are the discrete bands of continuous spectrum in δB_E; where each band corresponds to different $k_{\parallel}$'s value. Note that $|\delta B_R| \ll |\delta B_E|$ and $\delta B_N \propto \delta B_{\parallel}$ is rather ~~[illegible]~~ weak. These features are consistent with our theoretical predictions and are, indeed, marvelous testimony to the success of ideal MHD theory in describing low-frequency macroscopic wave dynamics.