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First, take $m_e \to 0^+$ in the electron eqn of motion (i.e., electron is massless, no inertia), we have

$$\underset{\sim}{E} + \underset{\sim}{V} \times B / c = 0 .$$

This implies $\underset{\sim}{B} \cdot \underset{\sim}{E} = 0$; i.e. electrons are infinitely mobile along $\underset{\sim}{B}$ to short circuit any electric field parallel to $\underset{\sim}{B}$. Summing up the eqn. of motions of both ions and electrons, we find

$$\rho_i \frac{d\underset{\sim}{V}}{dt} = \underset{\sim}{J} \times \underset{\sim}{B} / c \quad ;$$

where we have applied quasi-neutrality $n_e = n_i$ to eliminate the electric field, and noted again the plasma is taken to be "cold". $\rho_i = n \cdot m_i$ and inertia is carried by ions. Quasi-neutrality also implies

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{J} = 0 .$$

Consider infinitesimally small (linear) perturbations from the following stationary equilibrium

$$\underset{\sim}{V_0} = 0 \quad \Rightarrow \quad \underset{\sim}{V} = \delta \underset{\sim}{V}$$

$$\underset{\sim}{E_0} = 0 \quad \Rightarrow \quad \underset{\sim}{E} = \delta \underset{\sim}{E}$$

$$\underset{\sim}{J_0} = 0 \quad \Rightarrow \quad \underset{\sim}{J} = \delta \underset{\sim}{J}$$

$$\underset{\sim}{B_0} \neq 0 \quad \Rightarrow \quad \underset{\sim}{B} = \underset{\sim}{B_0} + \delta \underset{\sim}{B} \quad , \; n_0 \neq 0 .$$

δ's are 0^+ linear quantities

Thus, $\delta \underset{\sim}{V} \times \underset{\sim}{B}_0 / c = -\delta \underset{\sim}{E} \Rightarrow \delta \underset{\sim}{V}_\perp = c\, \delta \underset{\sim}{E} \times \underset{\sim}{B}_0 / B_0^2$.

$$\rho_{i0} \frac{\partial}{\partial t} \delta \underset{\sim}{V} = \delta \underset{\sim}{J} \times \underset{\sim}{B}_0 / c \Rightarrow \delta \underset{\sim}{J}_\perp = -\frac{\rho_{i0} c}{B_0^2} (\partial_t \delta \underset{\sim}{V}_\perp) \times \underset{\sim}{B}_0 .$$

Here, $\perp$ denotes (perpendicular to $\underset{\sim}{B}_0$) component.

$$\therefore \quad \boxed{\delta \underset{\sim}{J}_\perp = \frac{\rho_{i0} c^2}{B_0^2} \frac{\partial}{\partial t} \delta \underset{\sim}{E}_\perp .}$$

$\boxed{\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{J} = 0}$ with $\delta \underset{\sim}{J} = \frac{\delta J_\parallel}{B_0} \underset{\sim}{B}_0 + \delta \underset{\sim}{J}_\perp$. $\parallel$ = parallel to $\underset{\sim}{B}_0$

$$\Rightarrow \boxed{\underset{\sim}{B}_0 \cdot \underset{\sim}{\nabla} \frac{\delta J_\parallel}{B_0} + \underset{\sim}{\nabla}_\perp \cdot \delta \underset{\sim}{J}_\perp = 0 .}$$

Adopt scalar and vector potentials,

$(\delta\phi, \delta \underset{\sim}{A})$ Coulomb gauge $\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{A} = 0$.

$$\delta \underset{\sim}{E} = -\underset{\sim}{\nabla} \delta\phi - \frac{1}{c} \partial_t \delta \underset{\sim}{A} .$$

Ampere's Law $\quad \nabla^2 \delta A_\parallel = -4\pi\, \delta J_\parallel / c$

Alfvén waves have phase velocity << speed of light
$\Rightarrow$ displacement current is negligible.

Furthermore, we assume

$$|k_\parallel| \ll |k_\perp| \quad \text{or} \quad \left| \frac{\underset{\sim}{B}_0}{B} \cdot \underset{\sim}{\nabla} \right| \ll |\nabla_\perp| ;$$

again in the spirit of minimize $|k_\parallel|$.

$$\Rightarrow \nabla^2 \simeq \nabla_\perp^2 , \quad \text{and} \quad \delta \underset{\sim}{E}_\perp \cong -\underset{\sim}{\nabla}_\perp \delta\phi$$

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$\nabla \cdot \delta \underset{\sim}{J} = 0 \Rightarrow$

$$c\, \underset{\sim}{B_0} \cdot \underset{\sim}{\nabla} \frac{\nabla_\perp^2 \delta A_\parallel}{B_0} - 4\pi\, \nabla_\perp \cdot \delta \underset{\sim}{J_\perp} = 0$$

$$\delta E_\parallel = 0 = \underset{\sim}{B_0} \cdot \delta \underset{\sim}{E} \Rightarrow \boxed{\underset{\sim}{B_0} \cdot \nabla \delta\phi = \frac{1}{c} \partial_t\, \underset{\sim}{B_0} \cdot \delta \underset{\sim}{A} = \frac{B_0}{c} \partial_t \delta A_\parallel}$$

Denote: $\underset{\sim}{B_0} \cdot \underset{\sim}{\nabla} = B_0 \frac{\partial}{\partial l}$

$$\Rightarrow \boxed{B_0 \frac{\partial}{\partial l} \frac{1}{B_0} \nabla_\perp^2 \frac{\partial}{\partial l} \delta\phi - \frac{1}{V_A^2} \frac{\partial^2}{\partial t^2} \nabla_\perp^2 \delta\phi = 0}$$

with $V_A^2 = \dfrac{B_0^2(r,\theta)}{4\pi \rho_{i0}(r)} = \dfrac{B_0^2}{4\pi \rho_{i0}(r)} \cdot \left(1 + \dfrac{r}{R_0}\cos\theta\right)^{-2}$.

◎ It thus suggest the following <u>general wave equation</u>

$$\boxed{\mathcal{L}\left(\omega, \frac{\partial}{\partial l}, \nabla_\perp^2, \cos\theta, r\right) \delta\phi(r,\theta,\xi) = 0,}$$

where we have let $\frac{\partial}{\partial t} = -i\omega$, etc.

Taking, with the translational invariance constraint,

$$\delta\phi(r,\theta,\xi) = e^{in\xi} \sum_m e^{-im\theta}\, \delta\Phi(nq-m)$$

$\Rightarrow$ we have

$$e^{in\xi} \sum_m e^{-im\theta}\, \mathcal{L}\left(\omega, ik_{\parallel nm}, \frac{\partial^2}{\partial r^2} - \frac{n^2 q^2}{r_m^2}, T_c, r_m\right) \delta\Phi(nq-m) = 0$$

with $T_c\, \delta\Phi(nq-m) = \frac{1}{2}\left[\delta\Phi(nq-m-1) + \delta\Phi(nq-m+1)\right]$.

p16 With trans. inv., the linear wave eqn becomes

$$\boxed{\mathcal{L}\left(\omega, ik_{\parallel nm}, \partial_r^2 - \frac{n^2 q^2}{r_m^2}, T_c, r_m\right) \delta\Phi(nq-m) = 0.}$$

$k_{\parallel, nm} = \frac{1}{qR}(nq-m)$. Note it is a differential-difference equation. difference operator is due to T_c.

This equation becomes more readily solvable if we Fourier transform $\delta\Phi(nq-m)$!!

- Defining $\delta\hat{\Phi}(\eta)$ = F.T. of $\delta\Phi(nq-m)$; i.e.

$$\delta\hat{\Phi}(\eta) = \int \delta\Phi(nq-m)\, e^{i(nq-m)\eta}\, d(nq-m)$$

$$\delta\Phi(nq-m) = \frac{1}{2\pi}\int \delta\hat{\Phi}(\eta)\, e^{-i\eta(nq-m)}\, d\eta$$

$$\Rightarrow i k_{\parallel nm}\, \delta\Phi(nq-m) = \frac{1}{qR}\, i(nq-m)\,\delta\Phi(nq-m) \xRightarrow{F.T.} \frac{1}{qR}\frac{\partial}{\partial\eta}\delta\hat{\Phi}(\eta)$$

$$\Rightarrow \left(\partial_r^2 - \frac{n^2q^2}{r_m^2}\right)\delta\Phi \xRightarrow{F.T.} \left[-\frac{n^2q^2}{r_m^2} + (-i\eta n q')^2\right]\delta\hat{\Phi}(\eta)$$

$$= -\frac{n^2q^2}{r_m^2}\left[1 + \hat{s}^2\eta^2\right]\delta\hat{\Phi}(\eta)$$

$$\Rightarrow T_c\, \delta\Phi(nq-m) \xRightarrow{F.T.} (\cos\eta)\,\delta\hat{\Phi}(\eta) \qquad \hat{s} = \frac{r_m q'}{q} \text{ magnetic shear}$$

$$\Rightarrow \boxed{\mathcal{L}\left[\omega, \partial_\eta, -\frac{n^2q^2}{r_m^2}(1+\hat{s}^2\eta^2), \cos\eta, r_m\right]\delta\hat{\Phi}(\eta) = 0} \quad !!$$

ordinary differential eqn. parameterized by r_m.

We need to impose appropriate boundary conditions to uniquely specify the above linear eigenmode equation. One of the most useful consideration is taking the initial-value view point; i.e., the causality constraint. That is, $\delta\phi(t) \to 0$ as $t \to -\infty$. Taking $\delta\phi(t) \propto \delta\hat{\phi}(\eta)\exp(-i\omega t)$, we then dictate that

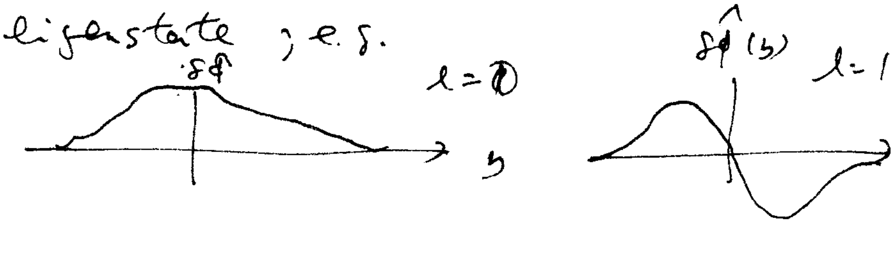
$$\delta\hat{\phi}(\eta) \to 0 \quad \text{as } |\eta| \to \infty \quad \text{for } \operatorname{Im}\omega > 0.$$

physically, one can understand this causality boundary condition by noting that, since the group velocity is finite, any initial perturbations $(t=0)$ with finite extent in η will not reach $|\eta| \to \infty$ during the finite t. Imposing the b.c.'s, we can, in principle, solve the eigenvalue and eigenfunction as

$$\omega = \omega(r_m \mid n, \ell) \qquad \ell = 0, 1, 2 ..$$

$$\delta\hat{\Phi} = \delta\hat{\Phi}(\eta \mid n, \ell, r_m) \quad ;$$

here ℓ is the ℓth eigenstate; e.g.



○ For the <u>shear Alfvén wave</u> discussed earlier, the eigenmode equation is

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$$\left\{ \frac{\partial}{\partial \eta} k_\perp^2(\eta) \frac{\partial}{\partial \eta} + \frac{\omega^2}{\omega_A^2(r_m)} k_\perp^2(\eta) \left[1 + 2\frac{r_m}{R_0} \cos(\eta) \right] \right\} \delta\phi(\eta) = 0;$$

where $$k_\perp^2 = \frac{n^2 q^2(r_m)}{r_m^2} \left[1 + \hat{s}^2(r_m)\, \eta^2 \right], \quad \omega_A^2 = \frac{V_A^2(r_m)}{q^2(r_m) R_0^2}.$$