drift-kinetic limit,

$|k_\perp \rho| \ll 1$, $\delta E_\parallel \approx 0$, $\delta B_\parallel \approx 0$

<u>ideal MHD</u> <u>$\beta \ll 1$</u>

$$\delta \hat{f} = -\frac{q}{m}\frac{\hat{\omega}_*}{\omega}\delta\hat{\phi}\, F_0 + \delta \hat{H}(\varepsilon, \mu, \ell)$$

$$(\partial_t + v_\parallel \partial_\ell + i\, \underset{\sim}{k_\perp} \cdot \underset{\sim}{v_d})\, \delta \hat{H} = i\frac{q}{m}\left(\omega\frac{\partial}{\partial\varepsilon} + \hat{\omega}_*\right) F_0 \frac{\omega_d}{\omega}\delta\hat{\phi}$$

$\underset{\sim}{v_d}$ = magnetic $\underset{\sim}{\nabla} B$ and curvature drift

$\varepsilon = v^2/2$, $\mu = v_\perp^2/2B$

$\underset{\sim}{k_\perp} \cdot \underset{\sim}{v_d} = \omega_d$,

$$\hat{\omega}_* F_0 = \frac{\underset{\sim}{k_\perp} \times \underset{\sim}{e_\parallel}}{\Omega} \cdot \underset{\sim}{\nabla} F_0 \quad = \text{diamagnetic drift}$$

We then have

$$\partial_t \rho \underset{\sim}{u} + \underset{\sim}{\nabla} \cdot \underset{\approx}{P} - \underset{\sim}{J} \times \underset{\sim}{B}/c = 0; \qquad (3)$$

where $\rho \underset{\sim}{u} = \sum_{j=e,i} (mn\underset{\sim}{u})_j$, $\underset{\sim}{J} = \sum_j q_j n_j \underset{\sim}{u}_j$ is the current density.

Linearize Eq. (3) w.r.t. $\rho = \rho_0 + \delta\rho$, $\underset{\sim}{u} = \delta \underset{\sim}{u}$, $\underset{\sim}{J} = \underset{\sim}{J}_0 + \delta \underset{\sim}{J}$, $\underset{\approx}{P} = P_0 \underset{\approx}{I} + \delta \underset{\approx}{P}$ (i.e. equilibrium pressure isotropic), we have

$$\partial_t \rho_0 \delta \underset{\sim}{u} + \underset{\sim}{\nabla} \cdot \delta \underset{\approx}{P} - \frac{1}{c}(\delta \underset{\sim}{J} \times \underset{\sim}{B}_0 + \underset{\sim}{J}_0 \times \delta \underset{\sim}{B}) = 0. \qquad (4)$$

Letting $\delta \underset{\sim}{u} = \partial_t \delta \underset{\sim}{\xi}$ and $\delta \underset{\sim}{\xi} = \delta \hat{\underset{\sim}{\xi}} e^{-i\omega t}$, we have

$$\boxed{\omega^2 \rho_0 \delta \hat{\underset{\sim}{\xi}} = \underset{\sim}{\nabla} \cdot \delta \hat{\underset{\approx}{P}} - \frac{1}{c}(\delta \hat{\underset{\sim}{J}} \times \underset{\sim}{B}_0 + \underset{\sim}{J}_0 \times \delta \hat{\underset{\sim}{B}}).} \qquad (5)$$

Now $\delta \hat{\underset{\approx}{P}}_j = m_j \int \underset{\sim}{v}\underset{\sim}{v}\, \delta \hat{f}_j \, d^3\underset{\sim}{v}$, and, from earlier notes,

$$\delta \hat{f}_j = -\delta \hat{\underset{\sim}{\xi}} \cdot \underset{\sim}{\nabla} F_{0j} + \delta \hat{H}_{tj} \qquad t: \text{trapped}.$$

$\delta \hat{H}$ circulating $= 0$.

Hence,

$$\delta \hat{\underset{\approx}{P}}_j = -(\delta \hat{\underset{\sim}{\xi}}_\perp \cdot \underset{\sim}{\nabla} P_0)\underset{\approx}{I} + \sum_j m_j \langle \underset{\sim}{v}\underset{\sim}{v}\, \delta \hat{H}_{tj} \rangle_v$$

$$\boxed{\equiv \delta \hat{P}_{conv} \underset{\approx}{I} + \delta \hat{\underset{\approx}{P}}_{comp,t}} \qquad (6)$$

Conv = convective , Comp = compressional

Now, since $\int d^3x = \int d\alpha \int v_\perp dv_\perp d\psi_1$ and $\frac{\partial \delta \hat{H}_t}{\partial \alpha} = 0$, thus,

it is easy to see that $\delta \hat{\underset{\sim}{P}}_{comp}$ is diagonal, i.e.,

$$\delta \hat{\underset{\sim}{P}}_{comp,t} = \begin{bmatrix} \delta\tilde{P}_\perp & & 0 \\ & \delta\tilde{P}_\perp & \\ 0 & & \delta\tilde{P}_\parallel \end{bmatrix} = \delta\tilde{P}_\perp \underset{=}{I} + (\delta\tilde{P}_\parallel - \delta\tilde{P}_\perp)\underset{\sim}{e}_\parallel \underset{\sim}{e}_\parallel$$

$\odot \frac{1}{2}\int \delta\hat{\underset{\sim}{\xi}}_\perp d^3x \cdot$ (Eq.)(5)

$$\Rightarrow \boxed{\omega^2 \delta\hat{I} = \delta\hat{W}_f + \delta\hat{W}_k ,} \text{ where} \qquad (7)$$

- $\delta\hat{I} = \frac{1}{2}\int |\delta\hat{\underset{\sim}{\xi}}_\perp^*|^2 \rho_0 \, d^3x$
- $\delta\hat{W}_f = \frac{1}{2}\int d^3\underset{\sim}{x} \; \delta\hat{\underset{\sim}{\xi}}_\perp^* \cdot [\underset{\sim}{\nabla} \delta\hat{P}_{conv} - \frac{1}{c}(\delta\hat{\underset{\sim}{J}} \times \underset{\sim}{B}_0 + \underset{\sim}{J}_0 \times \delta\hat{\underset{\sim}{B}})]$

Note that $\delta\hat{P}$, $\delta\hat{\underset{\sim}{J}} = \underset{\sim}{\nabla} \times \delta\hat{\underset{\sim}{B}}$, $\delta\hat{\underset{\sim}{B}} = \underset{\sim}{\nabla} \times [\delta\hat{\underset{\sim}{\xi}}_\perp \times \underset{\sim}{B}_0]$ are all functional of $\delta\hat{\underset{\sim}{\xi}}_\perp$. Thus, $\delta\hat{W}_f = \delta\hat{W}_f[\delta\hat{\underset{\sim}{\xi}}_\perp]$. [In fact, one can prove that $\delta\hat{W}_f$ is self-adjoint. One can identify it as the incompressible limit of the ideal MHD energy principle of Berstein et al, 1958]

- Finally,

p. 4

$$\delta \hat{W}_k = \frac{1}{2}\int d^3x\, \delta \hat{\xi}_\perp^{*} \cdot \left[\nabla \cdot \delta \hat{\mathbf{P}}_{comp,t} \right]$$

$$\nabla \cdot \delta \hat{\mathbf{P}}_{comp,t} = \nabla \cdot \left[\delta \widetilde{P}_\perp \mathbf{I} + (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp)\, \mathbf{e}_\parallel \mathbf{e}_\parallel \right]$$

$$= \nabla \delta \widetilde{P}_\perp + \nabla \cdot (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp)\, \mathbf{e}_\parallel \mathbf{e}_\parallel$$

$$= \nabla \delta \widetilde{P}_\perp + \mathbf{e}_\parallel \left[\nabla \cdot (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp)\, \mathbf{e}_\parallel \right] + (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp) \underbrace{(\mathbf{e}_\parallel \cdot \nabla)\, \mathbf{e}_\parallel}_{\boldsymbol{\kappa}\,:\ \text{curvature}}$$

$$= \nabla \delta \widetilde{P}_\perp + (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp)\, \boldsymbol{\kappa} + \mathbf{e}_\parallel \left[\nabla \cdot (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp)\, \mathbf{e}_\parallel \right]$$

$$\Rightarrow \delta \hat{W}_k = \frac{1}{2}\int d^3\mathbf{x} \left[\delta \hat{\xi}_\perp^{*} \cdot \nabla \delta \widetilde{P}_\perp + (\boldsymbol{\kappa} \cdot \delta \hat{\xi}_\perp^{*})(\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp) \right]$$

integration by parts

$$= -\frac{1}{2}\int d^3\mathbf{x} \left[\delta \widetilde{P}_\perp \nabla \cdot \delta \hat{\xi}_\perp^{*} - (\delta \widetilde{P}_\parallel - \delta \widetilde{P}_\perp)(\delta \hat{\xi}_\perp^{*} \cdot \boldsymbol{\kappa}) \right]$$

as <u>given in p. 49</u>